

# On the $N$ -Solitons Solutions in the Novikov–Veselov Equation

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**Abstract.** We construct the  $N$ -solitons solution in the Novikov–Veselov equation from the extended Moutard transformation and the Pfaffian structure. Also, the corresponding wave functions are obtained explicitly. As a result, the property characterizing the  $N$ -solitons wave function is proved using the Pfaffian expansion. This property corresponding to the discrete scattering data for  $N$ -solitons solution is obtained in [arXiv:0912.2155] from the  $\bar{\partial}$ -dressing method.

*Key words:* Novikov–Veselov equation;  $N$ -solitons solutions; Pfaffian expansion; wave functions

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## 1 Introduction

The Novikov–Veselov equation [3, 9, 34, 41] is defined by

$$\begin{aligned} U_t &= \partial_z^3 U + \bar{\partial}_z U + 3\partial_z(VU) + 3\bar{\partial}_z(V^*U), \\ \bar{\partial}_z V &= \partial_z U, \quad \partial_z V^* = \bar{\partial}_z U. \end{aligned} \tag{1}$$

When  $z = \bar{z} = x$ , we get the famous KdV equation ( $U = \bar{U} = V = \bar{V}$ )

$$U_t = 2U_{xxx} + 12UU_x.$$

The equation (1) can be represented as the form of Manakov's triad [24]

$$H_t = [A, H] + BH,$$

where  $H$  is the two-dimension Schrödinger operator

$$H = \partial_z \bar{\partial}_z + U$$

and

$$A = \partial_z^3 + V\partial_z + \bar{\partial}_z^3 + \bar{V}\bar{\partial}_z, \quad B = V_z + \bar{V}_{\bar{z}}.$$

It is equivalent to the linear representation

$$H\psi = 0, \quad \partial_t \psi = A\psi. \tag{2}$$

We see that the Novikov–Veselov equation (1) preserves a class of the purely potential self-adjoint operators  $H$ . Here the pure potential means  $H$  has no external electric and magnetic fields. The periodic inverse spectral problem for the two-dimensional Schrödinger operator  $H$

was investigated in terms of the Riemann surfaces with some group of involutions and the corresponding Prym  $\Theta$ -functions [5, 10, 22, 27, 28, 33, 37]. On the other hand, it is known that the Novikov–Veselov hierarchy is a special reduction of the two-component BKP hierarchy [23, 36, 40] (and references therein). In [23], the authors showed that the Drinfeld–Sokolov hierarchy of D-type is a reduction of the two-component BKP hierarchy using two different types of pseudo-differential operators, which is different from Shiota’s point of view [37]. Also, in [26], it is shown that the Tzitzeica equation is a stationary symmetry of the Novikov–Veselov equation. Finally, it is worthwhile to notice that the Novikov–Veselov equation (1) is a special reduction of the Davey–Stewartson equation [20, 21].

Let  $H\psi = H\omega = 0$ . Then via the Moutard transformation [1, 29, 30, 31]

$$\begin{aligned} U(z, \bar{z}) &\longrightarrow \hat{U}(z, \bar{z}) = U(z, \bar{z}) + 2\partial\bar{\partial}\ln\omega, \\ \psi &\longrightarrow \theta = \frac{1}{\omega} \int (\psi\partial\omega - \omega\partial\psi)dz - (\psi\bar{\partial}\omega - \omega\bar{\partial}\psi)d\bar{z}, \end{aligned} \quad (3)$$

one can construct a new Schrödinger operator  $\hat{H} = \partial_z\bar{\partial}_z + \hat{U}$  and  $\hat{H}\theta = 0$ . We remark that the Moutard transformation (3) is utilized to construct the  $N$ -solitons solutions of the Tzitzeica equation [15].

The extended Moutard transformation was established such that  $\hat{U}(t, z, \bar{z})$  and  $\hat{V}(t, z, \bar{z})$  defined by [13, 25]

$$\hat{U}(t, z, \bar{z}) = U(t, z, \bar{z}) + 2\partial\bar{\partial}\ln W(\psi, \omega), \quad \hat{V}(t, z, \bar{z}) = V(t, z, \bar{z}) + 2\partial\bar{\partial}\ln W(\psi, \omega),$$

where the skew product (alternating bilinear form)  $W$  is defined by

$$\begin{aligned} W(\psi, \omega) = & \int (\psi\partial\omega - \omega\partial\psi)dz - (\psi\bar{\partial}\omega - \omega\bar{\partial}\psi)d\bar{z} + [\psi\partial^3\omega - \omega\partial^3\psi + \omega\bar{\partial}^3 - \psi\bar{\partial}^3\omega \\ & + 2(\partial^2\psi\partial\omega - \partial\psi\partial^2\omega) - 2(\bar{\partial}^2\psi\bar{\partial}\omega - \bar{\partial}\psi\bar{\partial}^2\omega) + 3V(\psi\partial\omega - \omega\partial\psi) \\ & - 3\bar{V}(\psi\bar{\partial}\omega - \omega\bar{\partial}\psi)]dt, \end{aligned} \quad (4)$$

will also satisfy the Novikov–Veselov equation.

In [2, 6, 7, 8], the rational solutions and line solitons of the Novikov–Veselov equation (1) are constructed by the  $\bar{\partial}$ -dressing method. To get these kinds of solutions, the scattering datum have to be delta-type and the reality of  $U$  also puts some extra constraints on them. In [39], the singular rational solutions are obtained using the extended Moutard transformation (4); however, the non-singular rational solutions are constructed in [4].

Next, we construct Pfaffian-type solutions. Given any  $N$  wave functions  $\psi_1, \psi_2, \psi_3, \dots, \psi_N$  (or their linear combinations) of (2) for fixed potential  $U(z, \bar{z}, t)$ , the  $N$ -step extended Moutard transformation can be obtained in the Pfaffian [1, 31] (also see [12, 35])

$$P(\psi_1, \psi_2, \psi_3, \dots, \psi_N) = \begin{cases} \text{Pf}(\psi_1, \psi_2, \psi_3, \dots, \psi_N) & \text{if } N \text{ even,} \\ \widetilde{\text{Pf}}(\psi_1, \psi_2, \psi_3, \dots, \psi_N) & \text{if } N \text{ odd,} \end{cases}$$

$$\text{Pf}(\psi_1, \psi_2, \psi_3, \dots, \psi_N) = \sum_{\sigma} \epsilon(\sigma) W_{\sigma_1\sigma_2} W_{\sigma_3\sigma_4} \cdots W_{\sigma_{N-1}\sigma_N}, \quad (5)$$

$$\widetilde{\text{Pf}}(\psi_1, \psi_2, \psi_3, \dots, \psi_N) = \sum_{\sigma} \epsilon(\sigma) W_{\sigma_1\sigma_2} W_{\sigma_3\sigma_4} \cdots W_{\sigma_{N-2}\sigma_{N-1}} \psi_{\sigma_N}, \quad (6)$$

where  $W_{\sigma_i\sigma_j} = W(\psi_{\sigma(i)}, \psi_{\sigma(j)})$  is defined by the skew product (4). The summations  $\sigma$  in (5) and (6) run from over the permutations of  $\{1, 2, 3, \dots, N\}$  such that  $\sigma_1 < \sigma_2, \sigma_3 < \sigma_4, \sigma_5 < \sigma_6, \dots$  and  $\sigma_1 < \sigma_3 < \sigma_5 < \sigma_7 < \dots$ , with  $\epsilon(\sigma) = 1$  for the even permutations and  $\epsilon(\sigma) = -1$  for the odd permutations. Then the solution  $U$  and  $V$  can be expressed as [1]

$$U = U_0 + 2\partial\bar{\partial}[\ln P(\psi_1, \psi_2, \psi_3, \dots, \psi_N)], \quad V = V_0 + 2\partial\bar{\partial}[\ln P(\psi_1, \psi_2, \psi_3, \dots, \psi_N)],$$

and the corresponding wave function is

$$\varphi = \frac{P(\psi_1, \psi_2, \psi_3, \dots, \psi_N, \vartheta)}{P(\psi_1, \psi_2, \psi_3, \dots, \psi_N)}, \quad (7)$$

where  $\vartheta$  is an arbitrary wave function different from  $\psi_1, \psi_2, \psi_3, \dots, \psi_N$ .

The paper is organized as follows. In Section 2, we obtain the  $N$ -solitons solutions using the extended Moutard transformation and the Pfaffian expansion. Several examples are given. In Section 3, the  $N$ -solitonic wave function is derived using (7) and the Pfaffian expansion. Section 4 is used to prove a special property to characterize the  $N$ -solitons wave function. Section 5 is devoted to the concluding remarks.

## 2 $N$ -solitons solutions

In this section, one uses successive iterations of the extended Moutard transformation (4) to construct  $N$ -solitons solutions.

To obtain the  $N$ -solitons solutions, we assume that  $V = 0$  in (1) and recall that  $\partial\bar{\partial} = \frac{1}{4}\Delta$ . One considers  $U = -\epsilon \neq 0$ , i.e.,

$$\partial\bar{\partial}\varphi = \epsilon\varphi, \quad \varphi_t = \varphi_{zzz} + \varphi_{\bar{z}\bar{z}\bar{z}}, \quad (8)$$

where  $\epsilon$  is non-zero real constant. The general solution of (8) can be expressed as

$$\varphi(z, \bar{z}, t) = \int_{\Gamma} e^{(i\lambda)z + (i\lambda)^3 t + \frac{\epsilon}{i\lambda}\bar{z} + \frac{\epsilon^3}{(i\lambda)^3}t} \nu(\lambda) d\lambda, \quad (9)$$

where  $\nu(\lambda)$  is an arbitrary distribution and  $\Gamma$  is an arbitrary path of integration such that the r.h.s. of (9) is well defined.

Next, using (5) and (9), one can construct the  $N$ -solitons solutions. Let's take  $\nu_m(\lambda) = \delta(\lambda - p_m)$  and  $\nu_n(\lambda) = a_n \delta(\lambda - q_n)$ , where  $p_m, a_n, q_n$  are complex numbers. Then one defines

$$\phi_m = \frac{\varphi(p_m)}{\sqrt{3}} = \frac{1}{\sqrt{3}} e^{F(p_m)}, \quad \psi_n = a_n \frac{\varphi(q_n)}{\sqrt{3}} = \frac{a_n}{\sqrt{3}} e^{F(q_n)},$$

where

$$F(\lambda) = (i\lambda)z + (i\lambda)^3 t + \frac{\epsilon}{i\lambda}\bar{z} + \frac{\epsilon^3}{(i\lambda)^3}t.$$

Then a direct calculation of the extended Moutard transformation (4) can yield

$$\begin{aligned} W(\phi_m, \psi_n) &= ia_n \frac{q_n - p_m}{q_n + p_m} e^{F(p_m) + F(q_n)}, & W(\phi_m, \phi_n) &= i \frac{p_n - p_m}{p_n + p_m} e^{F(p_m) + F(p_n)}, \\ W(\psi_m, \psi_n) &= ia_m a_n \frac{q_n - q_m}{q_n + q_m} e^{F(q_m) + F(q_n)}. \end{aligned} \quad (10)$$

The  $N$ -solitons solutions are defined by

$$U(z, \bar{z}, t) = -\epsilon + 2\partial\bar{\partial} \ln \tau_N(z, \bar{z}, t), \quad V(z, \bar{z}, t) = 2\partial\bar{\partial} \ln \tau_N(z, \bar{z}, t),$$

and then

$$U(z, \bar{z}, t) \rightarrow -\epsilon \quad \text{as} \quad z\bar{z} \rightarrow \infty,$$

where  $t$  is fixed. The  $\tau$ -functions are defined as follows. For simplicity, let's denote

$$W(p_m, q_n) = W(\phi_m, \psi_n), \quad W(p_m, p_n) = W(\phi_m, \phi_n), \quad W(q_m, q_n) = W(\psi_m, \psi_n), \quad (11)$$

and notice that  $F(-\lambda) = -F(\lambda)$ . The  $\tau_N$  is defined as

$$\tau_N(z, \bar{z}, t) = \text{Pf}(-p_1, q_1, -p_2, q_2, -p_3, q_3, \dots, -p_N, q_N), \quad (12)$$

where

$$\begin{aligned} (-p_m, -p_n) &= W(-p_m, -p_n), & (-p_m, q_n) &= W(-p_m, q_n) + \delta_{mn}, \\ (q_m, q_n) &= W(q_m, q_n). \end{aligned} \quad (13)$$

To get the expansion of (12), we use the following useful formula [14, 38]

$$\text{Pf}(\mathbf{A} + \mathbf{B}) = \sum_{r=0}^s \sum_{\alpha \in I_{2r}^m} (-1)^{|\alpha| - r} \text{Pf}(\mathbf{A}_\alpha) \text{Pf}(\mathbf{B}_{\alpha^c}), \quad (14)$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are  $m \times m$  matrices and  $s = [m/2]$  is the integer part of  $m/2$ ; moreover, we denote by  $\alpha^c$  the complementary set of  $\alpha$  in the subset  $\{1, 2, 3, \dots, m\}$  which is arranged in increasing order, and  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_{2r}$  for  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{2r})$ . For the case (12), one has

$$\mathbf{A}_N(z, \bar{z}, t) = \begin{bmatrix} 0 & (-p_1, q_1) & (-p_1, -p_2) & (-p_1, q_2) & \cdots & (-p_1, q_N) \\ (q_1, -p_1) & 0 & (q_1, -p_2) & (q_1, q_2) & \cdots & (q_1, q_N) \\ (-p_2, -p_1) & (-p_2, q_1) & 0 & (-p_2, q_2) & \cdots & (-p_2, q_N) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ (q_N, -p_1) & (q_N, q_1) & (q_N, -p_2) & (q_N, q_2) & \cdots & 0 \end{bmatrix}$$

and

$$\mathbf{B}_N = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix},$$

where  $\mathbf{A}_N$  and  $\mathbf{B}_N$  are  $2N \times 2N$  matrices. Hence by (14) one can have the expansion of (12), i.e.,

$$\tau_N = 1 + \sum_{\ell=1}^N f_\ell + \sum_{m=2}^N \left( \sum_{1 \leq \ell_1 < \ell_2 < \dots < \ell_m \leq N} f_{\ell_1} f_{\ell_2} \cdots f_{\ell_m} \prod_{1 \leq j < k \leq m} \mathbb{P}_{\ell_j \ell_k} \right), \quad (15)$$

where

$$f_\ell = i a_\ell \frac{p_\ell + q_\ell}{q_\ell - p_\ell} e^{F(q_\ell) - F(p_\ell)}, \quad \mathbb{P}_{\ell_j \ell_k} = \frac{(p_{\ell_j} - p_{\ell_k})(q_{\ell_j} - q_{\ell_k})(p_{\ell_j} + q_{\ell_k})(q_{\ell_j} + p_{\ell_k})}{(p_{\ell_j} + p_{\ell_k})(q_{\ell_j} + q_{\ell_k})(p_{\ell_j} - q_{\ell_k})(q_{\ell_j} - p_{\ell_k})}.$$

Here we have utilized the formula that if  $\mathbf{C}$  is a  $2N \times 2N$  matrix with  $(i, j)$ -th entry  $\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j}$ , then one has the Schur identity [32, 36]

$$\text{Pf}(\mathbf{C}) = \prod_{1 \leq i < j \leq 2N} \left( \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right). \quad (16)$$

Next, we illustrate the formula (15) (or 12) with several examples.

(1) One-soliton solution:

$$\mathbf{A}_1(z, \bar{z}, t) = \begin{bmatrix} 0 & (-p_1, q_1) \\ (q_1, -p_1) & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$\tau_1 = \text{Pf}(\mathbf{A}_1 + \mathbf{B}_1) = 1 + ia_1 \frac{q_1 + p_1}{q_1 - p_1} e^{F(q_1) - F(p_1)}.$$

(2) Two-solitons solution:

$$\mathbf{A}_2(z, \bar{z}, t) = \begin{bmatrix} 0 & (-p_1, q_1) & (-p_1, -p_2) & (-p_1, q_2) \\ (q_1, -p_1) & 0 & (q_1, -p_2) & (q_1, q_2) \\ (-p_2, -p_1) & (-p_2, q_1) & 0 & (-p_2, q_2) \\ (q_2, -p_1) & (q_2, q_1) & (q_2, -p_2) & 0 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \tau_2 = \text{Pf}(\mathbf{A}_2 + \mathbf{B}_2) &= 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} e^{F(q_1) - F(p_1)} + ia_2 \frac{p_2 + q_2}{q_2 - p_2} e^{F(q_2) - F(p_2)} \\ &+ ia_1 ia_2 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 - p_1}{p_2 + p_1} \frac{q_2 - q_1}{q_2 + q_1} \frac{p_1 + q_2}{q_2 - p_1} \frac{p_2 + q_1}{p_2 - q_1} e^{F(q_1) - F(p_1) + F(q_2) - F(p_2)} \end{aligned}$$

or

$$\tau_2 = 1 + f_1 + f_2 + \mathbb{P}_{12} f_1 f_2, \tag{17}$$

where

$$f_1 = ia_1 \frac{p_1 + q_1}{q_1 - p_1} e^{F(q_1) - F(p_1)}, \quad f_2 = ia_2 \frac{p_2 + q_2}{q_2 - p_2} e^{F(q_2) - F(p_2)},$$

$$\mathbb{P}_{12} = \frac{p_1 - p_2}{p_1 + p_2} \frac{q_1 - q_2}{q_1 + q_2} \frac{p_1 + q_2}{p_1 - q_2} \frac{q_1 + p_2}{q_1 - p_2}.$$

The  $\tau_2$  soliton (17) is also found in [2] using the  $\bar{\partial}$ -dressing method.

(3) Three-solitons solution:

$$\mathbf{A}_3(z, \bar{z}, t) = \begin{bmatrix} 0 & (-p_1, q_1) & (-p_1, -p_2) & (-p_1, q_2) & (-p_1, -p_3) & (-p_1, q_3) \\ (q_1, -p_1) & 0 & (q_1, -p_2) & (q_1, q_2) & (q_1, -p_3) & (q_1, q_3) \\ (-p_2, -p_1) & (-p_2, q_1) & 0 & (-p_2, q_2) & (-p_2, -p_3) & (-p_2, q_3) \\ (q_2, -p_1) & (q_2, q_1) & (q_2, -p_2) & 0 & (q_2, -p_3) & (q_2, q_3) \\ (-p_3, -p_1) & (-p_3, q_1) & (-p_3, -p_2) & (-p_3, q_2) & 0 & (-p_3, q_3) \\ (q_3, -p_1) & (q_3, q_1) & (q_3, -p_2) & (q_3, q_2) & (q_3, -p_3) & 0 \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned}\tau_3 &= \text{Pf}(\mathbf{A}_3 + \mathbf{B}_3) \\ &= 1 + f_1 + f_2 + f_3 + \mathbb{P}_{12}f_1f_2 + \mathbb{P}_{13}f_1f_3 + \mathbb{P}_{23}f_2f_3 + \mathbb{P}_{12}\mathbb{P}_{13}\mathbb{P}_{23}f_1f_2f_3,\end{aligned}\quad (18)$$

where

$$\begin{aligned}f_1 &= ia_1 \frac{p_1 + q_1}{q_1 - p_1} e^{F(q_1) - F(p_1)}, & f_2 &= ia_2 \frac{p_2 + q_2}{q_2 - p_2} e^{F(q_2) - F(p_2)}, \\ f_3 &= ia_3 \frac{p_3 + q_3}{q_3 - p_3} e^{F(q_3) - F(p_3)}, & \mathbb{P}_{12} &= \frac{p_1 - p_2}{p_1 + p_2} \frac{q_1 - q_2}{q_1 + q_2} \frac{p_1 + q_2}{p_1 - q_2} \frac{q_1 + p_2}{q_1 - p_2}, \\ \mathbb{P}_{13} &= \frac{p_1 + q_3}{p_1 - q_3} \frac{q_1 + p_3}{q_1 - p_3} \frac{p_1 - p_3}{p_1 + p_3} \frac{q_1 - q_3}{q_1 + q_3}, & \mathbb{P}_{23} &= \frac{p_2 + q_3}{p_2 - q_3} \frac{q_2 + p_3}{q_2 - p_3} \frac{p_2 - p_3}{p_2 + p_3} \frac{q_2 - q_3}{q_2 + q_3}.\end{aligned}$$

### 3 The wave functions

In this section, one uses (7) to construct the corresponding wave function of the  $\tau$  function (15).

From (7), one knows that the corresponding wave function of the  $N$ -solitons (15) can be written as

$$\varphi_N = \frac{P\left(\frac{\varphi(-p_1)}{\sqrt{3}}, \frac{\varphi(q_1)}{\sqrt{3}}, \frac{\varphi(-p_2)}{\sqrt{3}}, \frac{\varphi(q_2)}{\sqrt{3}}, \dots, \frac{\varphi(-p_N)}{\sqrt{3}}, \frac{\varphi(q_N)}{\sqrt{3}}, \frac{\varphi(\lambda)}{\sqrt{3}}\right)}{\tau_N}.$$

Using the notations in (11), (12) and (13), we can express  $\varphi_N$  as

$$\varphi_N = \frac{P(-p_1, q_1, -p_2, q_2, \dots, -p_N, q_N, \lambda)}{\tau_N}. \quad (19)$$

But we notice that

$$(-p_m, \lambda)^\# = W(-p_m, \lambda), \quad (q_m, \lambda)^\# = W(q_m, \lambda), \quad (20)$$

where  $(\cdot)^\#$  means there is no  $\delta_{mn}$  here when compared with (13). Now, let's compute  $P(-p_1, q_1, -p_2, q_2, \dots, -p_N, q_N, \lambda)$  using (14). In this case,

$$P(-p_1, q_1, -p_2, q_2, \dots, -p_N, q_N, \lambda) = \text{Pf}(M_N + Q_N),$$

where

$$M_N = \begin{bmatrix} & & & & (-p_1, \lambda) & \frac{\varphi(-p_1)}{\sqrt{3}} \\ & & & & (q_1, \lambda) & \frac{\varphi(q_1)}{\sqrt{3}} \\ & & A_N(z, \bar{z}, t) & & \vdots & \vdots \\ & & & & (q_N, \lambda) & \frac{\varphi(q_N)}{\sqrt{3}} \\ (\lambda, -p_1) & (\lambda, q_1) & \cdots & (\lambda, q_N) & 0 & \frac{\varphi(\lambda)}{\sqrt{3}} \\ -\frac{\varphi(-p_1)}{\sqrt{3}} & -\frac{\varphi(q_1)}{\sqrt{3}} & \cdots & -\frac{\varphi(q_N)}{\sqrt{3}} & -\frac{\varphi(\lambda)}{\sqrt{3}} & 0 \end{bmatrix}$$

and

$$Q_N = \begin{bmatrix} & & 0 & 0 \\ & & 0 & 0 \\ & B_N & \vdots & \vdots \\ & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Using (16), a simple calculation yields

$$\begin{aligned} & \text{Pf}(M_N + Q_N) \\ &= \phi \left[ 1 + \sum_{\ell=1}^N h_{\ell}(\lambda) + \sum_{m=2}^N \left( \sum_{1 \leq \ell_1 < \ell_2 < \ell_3 < \dots < \ell_m \leq N} h_{\ell_1} h_{\ell_2} \dots h_{\ell_m} \prod_{1 \leq j < k \leq m} \mathbb{P}_{\ell_j \ell_k} \right) \right] \\ &= \phi \hat{\chi}_N(\lambda), \end{aligned} \quad (21)$$

where

$$\begin{aligned} \phi &= \frac{\varphi(\lambda)}{\sqrt{3}}, \quad h_{\ell}(\lambda) = ia_{\ell} \frac{p_{\ell} + q_{\ell}}{q_{\ell} - p_{\ell}} \frac{p_{\ell} + \lambda}{p_{\ell} - \lambda} \frac{q_{\ell} - \lambda}{q_{\ell} + \lambda} e^{F(q_{\ell}) - F(p_{\ell})} = f_{\ell} \frac{p_{\ell} + \lambda}{p_{\ell} - \lambda} \frac{q_{\ell} - \lambda}{q_{\ell} + \lambda}, \\ \hat{\chi}_N(\lambda) &= 1 + \sum_{\ell=1}^N h_{\ell}(\lambda) + \sum_{m=2}^N \left( \sum_{1 \leq \ell_1 < \ell_2 < \ell_3 < \dots < \ell_m \leq N} h_{\ell_1} h_{\ell_2} \dots h_{\ell_m} \prod_{1 \leq j < k \leq m} \mathbb{P}_{\ell_j \ell_k} \right), \end{aligned}$$

and  $\mathbb{P}_{\ell_j \ell_k}$  is defined in (15).

We give several examples here.

(1) The one-soliton wave function:

$$M_1 = \begin{bmatrix} A_1(z, \bar{z}, t) & (-p_1, \lambda) & \frac{\varphi(-p_1)}{\sqrt{3}} \\ & (q_1, \lambda) & \frac{\varphi(q_1)}{\sqrt{3}} \\ (\lambda, -p_1) & (\lambda, q_1) & 0 & \frac{\varphi(\lambda)}{\sqrt{3}} \\ -\frac{\varphi(-p_1)}{\sqrt{3}} & -\frac{\varphi(q_1)}{\sqrt{3}} & -\frac{\varphi(\lambda)}{\sqrt{3}} & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\varphi_1 = \frac{P(-p_1, q_1, \lambda)}{\tau_1} = \frac{\text{Pf}(M_1 + Q_1)}{\tau_1} = \frac{\phi}{\tau_1} \left( 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 + \lambda}{p_1 - \lambda} \frac{q_1 - \lambda}{q_1 + \lambda} e^{F(q_1) - F(p_1)} \right).$$

We remark that

$$\begin{aligned} \varphi_1 &= \frac{\phi}{1 + f_1} \left[ 1 + ia_1 \frac{p_1 + \lambda}{p_1 - \lambda} \frac{q_1 - \lambda}{q_1 + \lambda} f_1 \right] = \frac{\phi}{1 + f_1} \left[ 1 + ia_1 \left( \frac{2p_1}{p_1 - \lambda} - 1 \right) \left( \frac{2q_1}{q_1 + \lambda} - 1 \right) f_1 \right] \\ &= \phi \left[ \frac{(1 + f_1) + 2ia_1 \left( \frac{p_1}{p_1 - \lambda} - \frac{q_1}{q_1 + \lambda} \right) e^{F(q_1) - F(p_1)}}{1 + f_1} \right] \\ &= \phi \left[ 1 - 2ia_1 \left( \frac{p_1}{\lambda - p_1} + \frac{q_1}{\lambda + q_1} \right) \frac{e^{F(q_1) - F(p_1)}}{\tau_1} \right]. \end{aligned} \quad (22)$$

This is the one-soliton wave function in [2, p. 9].

(2) The two-soliton wave function:

$$M_2 = \begin{bmatrix} & & & & (-p_1, \lambda) & \frac{\varphi(-p_1)}{\sqrt{3}} \\ & & & & (q_1, \lambda) & \frac{\varphi(q_1)}{\sqrt{3}} \\ & & A_2(z, \bar{z}, t) & & (-p_2, \lambda) & \frac{\varphi(-p_2)}{\sqrt{3}} \\ & & & & (q_2, \lambda) & \frac{\varphi(q_2)}{\sqrt{3}} \\ (\lambda, -p_1) & (\lambda, q_1) & (\lambda, -p_2) & (\lambda, q_2) & 0 & \frac{\varphi(\lambda)}{\sqrt{3}} \\ -\frac{\varphi(-p_1)}{\sqrt{3}} & -\frac{\varphi(q_1)}{\sqrt{3}} & -\frac{\varphi(-p_2)}{\sqrt{3}} & -\frac{\varphi(q_2)}{\sqrt{3}} & -\frac{\varphi(\lambda)}{\sqrt{3}} & 0 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} & & 0 & 0 \\ & B_2 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then using (21), one has

$$\begin{aligned} \varphi_2 &= \frac{P(-p_1, q_1, -p_2, q_2, \lambda)}{\tau_2} = \frac{\text{Pf}(M_2 + Q_2)}{\tau_2} = \frac{\phi}{\tau_2} [1 + h_1(\lambda) + h_2(\lambda) + \mathbb{P}_{12} h_1(\lambda) h_2(\lambda)] \\ &= \frac{\phi}{\tau_2} \left( 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 + \lambda}{p_1 - \lambda} \frac{q_1 - \lambda}{q_1 + \lambda} e^{F(q_1) - F(p_1)} + ia_2 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 + \lambda}{p_2 - \lambda} \frac{q_2 - \lambda}{q_2 + \lambda} e^{F(q_2) - F(p_2)} \right. \\ &\quad \left. + ia_1 ia_2 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 + \lambda}{p_1 - \lambda} \frac{q_1 - \lambda}{q_1 + \lambda} \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 + \lambda}{p_2 - \lambda} \frac{q_2 - \lambda}{q_2 + \lambda} e^{F(q_1) + F(q_2) - F(p_1) - F(p_2)} \right). \end{aligned}$$

(3) Three-solitons wave function:

$$M_3 = \begin{bmatrix} & & & & & & (-p_1, \lambda) & \frac{\varphi(-p_1)}{\sqrt{3}} \\ & & & & & & (q_1, \lambda) & \frac{\varphi(q_1)}{\sqrt{3}} \\ & & & & & & (-p_2, \lambda) & \frac{\varphi(-p_2)}{\sqrt{3}} \\ & & & & & & (q_2, \lambda) & \frac{\varphi(q_2)}{\sqrt{3}} \\ & & & & & & (-p_3, \lambda) & \frac{\varphi(-p_3)}{\sqrt{3}} \\ & & & & & & (q_3, \lambda) & \frac{\varphi(q_3)}{\sqrt{3}} \\ (\lambda, -p_1) & (\lambda, q_1) & (\lambda, -p_2) & (\lambda, q_2) & (\lambda, -p_3) & (\lambda, q_3) & 0 & \frac{\varphi(\lambda)}{\sqrt{3}} \\ -\frac{\varphi(-p_1)}{\sqrt{3}} & -\frac{\varphi(q_1)}{\sqrt{3}} & -\frac{\varphi(-p_2)}{\sqrt{3}} & -\frac{\varphi(q_2)}{\sqrt{3}} & -\frac{\varphi(-p_3)}{\sqrt{3}} & -\frac{\varphi(q_3)}{\sqrt{3}} & -\frac{\varphi(\lambda)}{\sqrt{3}} & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & B_3 & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From (21), we get

$$\begin{aligned} \varphi_3 &= \frac{P(-p_1, q_1, -p_2, q_2, -p_3, q_3, \lambda)}{\tau_3} = \frac{\text{Pf}(M_3 + Q_3)}{\tau_3} \\ &= \frac{\phi}{\tau_3} [1 + h_1(\lambda) + h_2(\lambda) + h_3(\lambda) + \mathbb{P}_{12} h_1(\lambda) h_2(\lambda) + \mathbb{P}_{13} h_1(\lambda) h_3(\lambda) + \mathbb{P}_{23} h_2(\lambda) h_3(\lambda) \\ &\quad + \mathbb{P}_{12} \mathbb{P}_{13} \mathbb{P}_{23} h_1(\lambda) h_2(\lambda) h_3(\lambda)], \end{aligned}$$

where

$$\begin{aligned} h_1 &= ia_1 \frac{p_1 + \lambda}{p_1 - \lambda} \frac{q_1 - \lambda}{q_1 + \lambda} \frac{p_1 + q_1}{q_1 - p_1} e^{F(q_1) - F(p_1)}, & h_2 &= ia_2 \frac{p_2 + \lambda}{p_2 - \lambda} \frac{q_2 - \lambda}{q_2 + \lambda} \frac{p_2 + q_2}{q_2 - p_2} e^{F(q_2) - F(p_2)}, \\ h_3 &= ia_3 \frac{p_3 + \lambda}{p_3 - \lambda} \frac{q_3 - \lambda}{q_3 + \lambda} \frac{p_3 + q_3}{q_3 - p_3} e^{F(q_3) - F(p_3)} \end{aligned}$$

and  $\mathbb{P}_{12}$ ,  $\mathbb{P}_{13}$  and  $\mathbb{P}_{23}$  are defined in (18).



## 4 A property of $N$ -solitons wave function

In this section, we will express the wave function (19) as another form to generalize the equation (22) to  $N$ -solitons case.

Firstly, according to the Pfaffian expansion in [11], it is not difficult to see that

$$\begin{aligned} & \widetilde{\text{Pf}}(b_1, b_2, b_3, b_4, \dots, b_{2n-1}, b_{2n}, b_{2n+1}) \\ &= \sum_{m=1}^{2n+1} (-1)^{j+m} (b_j, b_m) \widetilde{\text{Pf}}(b_1, b_2, \dots, \hat{b}_j, \dots, \hat{b}_m, \dots, b_{2n}, b_{2n+1}), \\ & \text{for } j = 1, 2, \dots, 2n+1, \end{aligned} \quad (23)$$

where  $\hat{b}_j$  and  $\hat{b}_m$  mean these two terms are omitted.

Secondly, noticing (10) and letting  $\lambda = -p_\alpha$  or  $\lambda = q_\alpha$ ,  $\alpha = 1, 2, \dots, n$ , we have

$$\begin{aligned} & \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \dots, -p_N, q_N, -p_\alpha) \\ &= \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \dots, -p_N, q_N) = \phi(-p_\alpha) \hat{\chi}_N(-p_\alpha), \\ & \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, q_{\alpha-1}, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \dots, -p_N, q_N, q_\alpha) \\ &= \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, q_{\alpha-1}, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \dots, -p_N, q_N) = a_\alpha \phi(q_\alpha) \hat{\chi}_N(q_\alpha). \end{aligned} \quad (24)$$

They can be seen as follows. By (23), one has

$$\begin{aligned} & \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \dots, -p_N, q_N, -p_\alpha) \\ &= -(q_\alpha, -p_1) \widetilde{\text{Pf}}(q_1, -p_2, q_2, \dots, -p_{\alpha-1}, q_{\alpha-1}, -p_\alpha, -p_{\alpha+1}, \dots, -p_N, q_N, -p_\alpha) \\ & \quad + (q_\alpha, q_1) \widetilde{\text{Pf}}(-p_1, -p_2, q_2, \dots, -p_{\alpha-1}, q_{\alpha-1}, -p_\alpha, -p_{\alpha+1}, \dots, -p_N, q_N, -p_\alpha) - \dots \\ & \quad - (q_\alpha, -p_\alpha) \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_{\alpha-1}, q_{\alpha-1}, -p_{\alpha+1}, \dots, -p_N, q_N, -p_\alpha) + \dots \\ & \quad + (q_\alpha, -p_\alpha)^\# \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_{\alpha-1}, q_{\alpha-1}, -p_\alpha, -p_{\alpha+1}, \dots, -p_N, q_N) \\ &= [-(q_\alpha, -p_\alpha) + (q_\alpha, -p_\alpha)^\#] \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_{\alpha-1}, q_{\alpha-1}, -p_\alpha, -p_{\alpha+1}, \dots, -p_N, q_N) \\ &= \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_{\alpha-1}, q_{\alpha-1}, -p_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \dots, -p_N, q_N), \end{aligned}$$

where  $(q_\alpha, -p_\alpha)^\#$  is defined in (20) and we know that

$$\widetilde{\text{Pf}}(\dots, -p_\alpha, \dots, -p_\alpha) = 0.$$

The second equation of (24) can be proved similarly.

Finally, from (23), one yields

$$\begin{aligned} & \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_N, q_N, \lambda) = (\lambda, -p_1) \widetilde{\text{Pf}}(q_1, -p_2, q_2, \dots, -p_N, q_N) \\ & \quad - (\lambda, q_1) \widetilde{\text{Pf}}(-p_1, -p_2, q_2, \dots, -p_N, q_N) + (\lambda, -p_2) \widetilde{\text{Pf}}(q_1, -p_1, q_2, \dots, -p_N, q_N) \\ & \quad - (\lambda, q_2) \widetilde{\text{Pf}}(q_1, -p_1, -p_2, \dots, -p_N, q_N) + \dots \\ & \quad + (\lambda, -p_N) \widetilde{\text{Pf}}(-p_1, -p_2, q_2, \dots, -p_{N-1}, q_{N-1}, q_N) \\ & \quad - (\lambda, q_N) \widetilde{\text{Pf}}(-p_1, -p_2, q_2, \dots, -p_{N-1}, q_{N-1}, -p_N) + \phi(\lambda) \tau_N. \end{aligned}$$

Also, the wave function (21) can be written as

$$\varphi_N(\lambda) = \phi(\lambda) \chi_N(\lambda),$$

where  $\chi_N(\lambda) = \frac{\hat{\chi}_N(\lambda)}{\tau_N}$ . Therefore, using (24) and letting  $\Delta F_n = F(q_n) - F(p_n)$ , we get

$$\begin{aligned}
\chi_N(\lambda) &= 1 - \frac{p_1 + \lambda}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_N(q_1) - \frac{q_1 - \lambda}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_N(-p_1) \\
&\quad - \frac{p_2 + \lambda}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_N(q_2) - \frac{q_2 - \lambda}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_N(-p_2) - \cdots \\
&\quad - \frac{p_N + \lambda}{\lambda - p_N} ia_N e^{\Delta F_N} \chi_N(q_N) - \frac{q_N - \lambda}{q_N + \lambda} ia_N e^{\Delta F_N} \chi_N(-p_N) \\
&= 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_N(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_N(-p_1) \\
&\quad - \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_N(q_2) - \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_N(-p_2) - \cdots \\
&\quad - \frac{2p_N}{\lambda - p_N} ia_N e^{\Delta F_N} \chi_N(q_N) - \frac{2q_N}{q_N + \lambda} ia_N e^{\Delta F_N} \chi_N(-p_N) \\
&\quad + [-ia_1 e^{\Delta F_1} \chi_N(q_1) + ia_1 e^{\Delta F_1} \chi_N(-p_1) - ia_2 e^{\Delta F_2} \chi_N(q_2) \\
&\quad + ia_2 e^{\Delta F_2} \chi_N(-p_2) - \cdots - ia_N e^{\Delta F_N} \chi_N(q_N) + ia_N e^{\Delta F_N} \chi_N(-p_N)]. \tag{25}
\end{aligned}$$

Since  $\widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \dots, -p_N, q_N, 0) = \frac{\tau_N}{\sqrt{3}}$ , we have  $\hat{\chi}_N(0) = \tau_N$ . Then the last term in  $[\dots]$  of (25) is zero (or  $\lim_{\lambda \rightarrow \infty} \chi_N(\lambda) = 1$ ). Hence one has

$$\begin{aligned}
\chi_N(\lambda) &= 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_N(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_N(-p_1) \\
&\quad - \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_N(q_2) - \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_N(-p_2) - \cdots \\
&\quad - \frac{2p_N}{\lambda - p_N} ia_N e^{\Delta F_N} \chi_N(q_N) - \frac{2q_N}{q_N + \lambda} ia_N e^{\Delta F_N} \chi_N(-p_N).
\end{aligned}$$

This formula is also obtained by the d-bar dressing method when the d-bar data is the degenerate delta kernel [2, p. 6].

When  $n=1$ , we have (22). For  $n=2$ , from (21), one knows that

$$\begin{aligned}
\chi_2(-p_1) &= \left[ 1 + ia_2 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 - p_1}{p_2 + p_1} \frac{q_2 + p_1}{q_2 - p_1} e^{\Delta F_2} \right] / \tau_2, \\
\chi_2(q_1) &= \left[ 1 + ia_2 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 + q_1}{p_2 - q_1} \frac{q_2 - q_1}{q_2 + q_1} e^{\Delta F_2} \right] / \tau_2, \\
\chi_2(-p_2) &= \left[ 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 - p_2}{p_1 + p_2} \frac{q_1 + p_2}{q_1 - p_2} e^{\Delta F_1} \right] / \tau_2, \\
\chi_2(q_2) &= \left[ 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 + q_2}{p_1 - q_2} \frac{q_1 - q_2}{q_1 + q_2} e^{\Delta F_1} \right] / \tau_2.
\end{aligned}$$

Then

$$\begin{aligned}
\chi_2(\lambda) &= 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_2(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_2(-p_1) \\
&\quad - \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_2(q_2) - \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_2(-p_2).
\end{aligned}$$

We remark that this formula also appears in [2, p. 10], the parameters being different. For  $n=3$ , by (21), we obtain

$$\chi_3(-p_1) = \left[ 1 + ia_2 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 - p_1}{p_2 + p_1} \frac{q_2 + p_1}{q_2 - p_1} e^{\Delta F_2} + ia_3 \frac{p_3 + q_3}{q_3 - p_3} \frac{p_3 - p_1}{p_3 + p_1} \frac{q_3 + p_1}{q_3 - p_1} e^{\Delta F_3} \right] / \tau_3$$

$$\begin{aligned}
& + ia_2 ia_3 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 - p_1}{p_2 + p_1} \frac{q_2 + q_1}{q_2 - q_1} \frac{p_3 + q_3}{p_3 - p_3} \frac{p_3 - p_1}{p_3 + p_1} \frac{q_3 + q_1}{q_3 - q_1} \mathbb{P}_{23} e^{\Delta F_2 + \Delta F_3} \Big] / \tau_3, \\
\chi_3(q_1) = & \left[ 1 + ia_2 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 + q_1}{p_2 - q_1} \frac{q_2 - q_1}{q_2 + q_1} e^{\Delta F_2} + ia_3 \frac{p_3 + q_3}{q_3 - p_3} \frac{p_2 + q_1}{p_2 - q_1} \frac{q_3 - q_1}{q_3 + q_1} e^{\Delta F_3} \right. \\
& + ia_2 ia_3 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 + q_1}{p_2 - q_1} \frac{q_2 - q_1}{q_2 + q_1} \frac{p_3 + q_3}{p_3 - p_3} \frac{p_2 + q_1}{p_2 - q_1} \frac{q_3 - q_1}{q_3 + q_1} \mathbb{P}_{23} e^{\Delta F_2 + \Delta F_3} \Big] / \tau_3, \\
\chi_3(-p_2) = & \left[ 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 - p_2}{p_1 + p_2} \frac{q_1 + q_2}{q_1 - q_2} e^{\Delta F_1} + ia_3 \frac{p_3 + q_3}{p_3 - p_3} \frac{p_3 - p_2}{p_3 + p_2} \frac{q_3 + q_2}{q_3 - q_2} e^{\Delta F_3} \right. \\
& + ia_1 ia_3 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 - p_2}{p_1 + p_2} \frac{q_1 + q_2}{q_1 - q_2} \frac{p_3 + q_3}{p_3 - p_3} \frac{p_3 - p_2}{p_3 + p_2} \frac{q_3 + q_2}{q_3 - q_2} \mathbb{P}_{13} e^{\Delta F_1 + \Delta F_3} \Big] / \tau_3, \\
\chi_3(q_2) = & \left[ 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 + q_2}{p_1 - q_2} \frac{q_1 - q_2}{q_1 + q_2} e^{\Delta F_1} + ia_3 \frac{p_3 + q_3}{q_3 - p_3} \frac{p_3 + q_2}{p_3 - q_2} \frac{q_3 - q_2}{q_3 + q_2} e^{\Delta F_3} \right. \\
& + ia_1 ia_3 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 + q_2}{p_1 - q_2} \frac{q_1 - q_2}{q_1 + q_2} \frac{p_3 + q_3}{q_3 - p_3} \frac{p_3 + q_2}{p_3 - q_2} \frac{q_3 - q_2}{q_3 + q_2} \mathbb{P}_{13} e^{\Delta F_1 + \Delta F_3} \Big] / \tau_3, \\
\chi_3(-p_3) = & \left[ 1 + ia_1 \frac{p_1 + q_1}{p_1 - q_1} \frac{p_1 - p_3}{p_1 + p_3} \frac{q_1 + q_3}{q_1 - q_3} e^{\Delta F_1} + ia_2 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 - p_3}{p_2 + p_3} \frac{q_2 + q_3}{q_2 - q_3} e^{\Delta F_2} \right. \\
& + ia_1 ia_2 \frac{p_1 + q_1}{p_1 - q_1} \frac{p_1 - p_3}{p_1 + p_3} \frac{q_1 + q_3}{q_1 - q_3} \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 - p_3}{p_2 + p_3} \frac{q_2 + q_3}{q_2 - q_3} \mathbb{P}_{12} e^{\Delta F_1 + \Delta F_2} \Big] / \tau_3, \\
\chi_3(q_3) = & \left[ 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 + q_3}{p_1 - q_3} \frac{q_1 - q_3}{q_1 + q_3} e^{\Delta F_1} + ia_2 \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 + q_3}{p_2 - q_3} \frac{q_2 - q_3}{q_2 + q_3} e^{\Delta F_2} \right. \\
& + ia_1 ia_2 \frac{p_1 + q_1}{q_1 - p_1} \frac{p_1 + q_3}{p_1 - q_3} \frac{q_1 - q_3}{q_1 + q_3} \frac{p_2 + q_2}{q_2 - p_2} \frac{p_2 + q_3}{p_2 - q_3} \frac{q_2 - q_3}{q_2 + q_3} \mathbb{P}_{12} e^{\Delta F_1 + \Delta F_2} \Big] / \tau_3,
\end{aligned}$$

where  $\mathbb{P}_{12}$ ,  $\mathbb{P}_{13}$  and  $\mathbb{P}_{23}$  are defined in (18). Then

$$\begin{aligned}
\chi_3(\lambda) = & 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_3(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_3(-p_1) - \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_3(q_2) \\
& - \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_3(-p_2) - \frac{2p_3}{\lambda - p_3} ia_3 e^{\Delta F_3} \chi_3(q_3) - \frac{2q_3}{q_3 + \lambda} ia_3 e^{\Delta F_3} \chi_3(-p_3).
\end{aligned}$$

## 5 Concluding remarks

In this paper, we have used the extended Moutard transformation to construct the  $N$ -solitons solutions. The basic idea comes from the successive iterations of solitons solutions, as remains to be the simple method to obtain the  $N$ -solitons solutions. Also, the corresponding wave functions are constructed by the Pfaffian expansion of the sum of two anti-symmetric matrices (14) when compared with the  $\bar{\partial}$ -dressing method [2].

To obtain real  $N$ -solitons solutions of the Novikov–Veselov equation (1), one has to put extra relations between  $-p_i$  and  $q_i$  [2]. It could be interesting to investigate these real solutions for the Schrödinger operator (self-adjoint). On the other hand, the resonance of  $N$ -solitons solutions of DKP or KP theory has been studied in [16, 17, 18, 19]. And then the resonance of  $N$ -solitons solutions of Pfaffian type (15) deserves to be investigated. These issues will be published elsewhere.

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